Illustrating Some Implications of the Conservation Laws in Relativistic Mechanics

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Abstract

The conservation laws of nonrelativistic and relativistic systems are reviewed and some simple illustrations are provided for the restrictive nature of the relativistic conservation law involving the center of energy compared to the nonrelativistic conservation law for the center of restmass. Extension of the nonrelativistic interaction of particles through a potential to a system which is Lorentz-invariant through order v^2/c^2 is found to require new velocity- and acceleration-dependent forces which are suggestive of field theory where the no-interaction theorem of Currie, Jordan, and Sudershan does not hold.

I.   INTRODUCTION

At the beginning of the twentieth century, the mismatch between mechanics and electromagnetism led to the creation of two new theories, quantum mechanics and special relativity. Today the mismatch between mechanics and relativity is still not appreciated by many students of physics.   There is a tendency to believe that one can pass from a nonrelativistic mechanical theory over to a relativistic theory simply by using relativistic expressions for the mechanical energy and momentum of a particle while retaining a general nonrelativistic potential.[1]   In 1963 Currie, Jordan, and Sudarshan[2] proved their "no-interaction theorem," pointing out just how wrong this point of view really is.   They proved that two point particles which satisfy the conservation laws of Lorentz-invariant mechanics simply can not interact except through point contact forces.   This no-interaction theorem is sometimes alluded to in mechanics texts as an afterthought in discussions of one-particle relativistic motion.[3]   However, the text books seem to contain no physical examples of what is involved in the theorem.   In the present article we wish to rectify this omission by reviewing the conservation laws associated with Lorentz invariance and then exploring some simple mechanical examples which suggest the need for a relativistic field theory such as classical electrodynamics.

We start by listing the conservation laws of physical systems when in the presence of external forces.   These laws are associated with momentum, energy, angular momentum, and center-of-energy motion.   It is the fourth law associated with the symmetry of systems under change of inertial frame which is so different between nonrelativistic and relativistic systems.   We emphasize that this fourth conservation law is linked to the continuous flow of energy in relativistic systems, and this continuous flow of energy places severe restrictions on allowed systems.   In the nonrelativistic limit, this fourth conservation law reduces to the continuous flow of rest mass, so that the fourth conservation law merely restates the law of momentum conservation for particles, and places no restrictions upon nonrelativistic mechanical systems.

As our first example, we consider an external force applied to a single particle and use the conservation laws to derive both the Galilean- and the Lorentz-invariant expressions for mechanical energy and momentum.   Second, we consider the collision of two point particles at a single point, and show that all the conservation laws of either Galilean- or

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Lorentz-invariant physics can be satisfied by such single-point collisions. If one considers the collisions of one particle which is described by Galilean-invariant mechanical expressions and one particle which is described by Lorentz-invariant mechanical expressions, then the conservation laws of momentum, energy, and angular momentum can still be satisfied in a fixed inertial frame, but the outcome of the collision will depend specifically on the choice of inertial frame; the fourth conservation law associated with Galilean or Lorentz invariance will not hold in any inertial frame. The third set of examples involves collisions of particles which do not interact at a single point but through a potential. We show that the continuous flow of energy does not hold if the particles interact through a rigid rod or through a general potential. The one case where the Lorentz-invariant conservation law can be found to hold involves a constant force between the particles independent of their separation, a situation which can be reinterpreted within familiar physics as the collision of two electrically charged capacitor plates. The fourth section discusses the modification of a general potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ by adding velocity-dependent terms so as to make the interaction of particles Lorentz-invariant through order v^2/c^2. Such a modification leads to velocity- and acceleration-dependent forces between the particles and suggests a field-theory interaction for Lorentz-invariant behavior. The Darwin Lagrangian is seen to follow as the approximately Lorentz-invariant extension from the Coulomb potential, and the Darwin Lagrangian is known to be the valid v^2/c^2 approximation to classical electrodynamics. Finally we present a closing summary and discussion.

II. THE CONSERVATION LAWS IN RELATIVISTIC AND NONRELATIVISTIC PHYSICS

In the usual formulations of mechanics and field theory, the conservation laws of physical systems are associated with the generators of symmetry transformations for the systems. These include linear momentum conservation associated with space-translation invariance, energy conservation associated with time-translation invariance, angular momentum conservation associated with rotation invariance, and finally two distinct conservation laws associated with invariance under transformation to a new inertial frame. The generator of Galilean transformations is the total system restmass times the center of mass, while the generator of Lorentz transformations is the total energy times the center of energy.[4]

In the presence of external forces $\mathbf{F}_{ext\,i}$ on the particles m_i located at positions $\mathbf{r}_i$ moving with velocity $\mathbf{v}_i$, the conservation laws for a Lorentz-invariant mechanical system or field theory take the following forms.[5] The sum of the external forces on the system gives the time rate of change of system momentum $\mathbf{P}$

$$\sum_i \mathbf{F}_{ext\,i} = \frac{d\mathbf{P}}{dt} \tag{1}$$

The total power delivered to the system by the external forces gives the time rate of change of system energy U

$$\sum_i \mathbf{F}_{ext\,i} \cdot \mathbf{v}_i = \frac{dU}{dt} \tag{2}$$

The sum of the external torques gives the time rate of change of system angular momentum $\mathbf{L}$

$$\sum_i \mathbf{r}_i \times \mathbf{F}_{ext\,i} = \frac{d\mathbf{L}}{dt} \tag{3}$$

The external-power-weighted position equals the time rate of change of the quantity (the total energy U times the center of energy $\vec{\mathcal{X}}_{energy}$) minus c^2 times the system momentum

$$\sum_i \left(\mathbf{F}_{ext\,i} \cdot \mathbf{v}_i \right) \mathbf{r}_i = \frac{d}{dt}(U\vec{\mathcal{X}}_{energy}) - c^2\mathbf{P} \tag{4}$$

Here the center of energy $\vec{\mathcal{X}}_{energy}$ is defined so that

$$U\vec{\mathcal{X}}_{energy} = \sum_i U_i\mathbf{r}_i + \int d^3r u(\mathbf{r})\mathbf{r} \tag{5}$$

where U_i is the mechanical energy of the ith particle and $u(\mathbf{r})$ is the continuous system energy density at position $\mathbf{r}$. This last conservation law (4) expresses the continuous flow of energy in Lorentz-invariant systems. In an isolated system where no external forces are present, the linear momentum, energy, and angular momentum are all constants in time, and the center of energy $\vec{\mathcal{X}}_{energy}$ moves with constant velocity $d\vec{\mathcal{X}}_{energy}/dt = c^2\mathbf{P}/U$ since the energy U and momentum $\mathbf{P}$ in Eq. (4) are both constant.

The conservation laws for Galilean invariance involve exactly the same expression in Eqs. (1)-(3) concerning linear momentum, energy, and angular momentum. However the last equation (4) involving the center of energy takes the degenerate form

$$0 = \frac{d}{dt}\left((\sum_i m_i)\vec{\mathcal{X}}_{mass} \right) - \mathbf{P} \tag{6}$$

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obtained by dividing Eq. (4) through by c^2 and taking the $c \to \infty$ limit. In this case, the particle energy U_i divided by c^2 becomes the rest mass, $U_i/c^2 \to m_i$ when we take the limit $c \to \infty$. We note that in this limit, $(\mathbf{F}_{ext\,i} \cdot \mathbf{v}_i)\mathbf{r}_i/c^2 \to 0$ so that the external forces do not enter this fourth (and last) Galilean conservation law. Within Galilean invariance, the fourth conservation law expresses the continuous flow of restmass and this continuity is not interrupted by the presence of external forces which may introduce linear momentum, energy, and angular momentum into the system.

III. DERIVATIONS OF EXPRESSIONS FOR MECHANICAL LINEAR MOMENTUM AND ENERGY OF A PARTICLE

We can use these conservation laws when applied to a single particle to derive the non-relativistic and relativistic expressions for mechanical energy and momentum. If a single particle of mass m experience a force $\mathbf{F}$, then according to the first conservation law in Eq. (1), the particle mechanical momentum $\mathbf{p}$ (which is the entire momentum of the system) changes as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{7}$$

while the change in mechanical energy U of the particle is given by the second conservation law in Eq. (2)

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{dU}{dt} \tag{8}$$

The third conservation law is not needed here for the present interest of exploring the linear momentum and energy of a point particle. The fourth conservation law takes a different form for Galilean and Lorentz invariance.

A. Galilean Invariance

The center of mass of a one-particle system is clearly located at the position $\mathbf{r}$ of the single particle. Then the fourth Galilean conservation law (6) for the center of mass of this one-particle system satisfies

$$\frac{d}{dt}(m\mathbf{r}) = \mathbf{p} \tag{9}$$

so that the mechanical momentum of the nonrelativistic particle is identified in terms of the time rate of change of the mass times the position of the particle. Then since the mass

m is a constant in time, the particle mechanical momentum must be given by the familiar nonrelativistic expression

$$m\frac{d\mathbf{r}}{dt} = \mathbf{p} \tag{10}$$

Then from Eqs. (7) and (8), $dU/dt = (d\mathbf{p}/dt) \cdot (d\mathbf{r}/dt)$, and then from Eq. (10) it follows that $U = \frac{1}{2}m\,(d\mathbf{r}/dt)^2 + const$ where the constant corresponds to an undetermined zero of energy. Usually for nonrelativistic mechanical energy, the constant is chosen to vanish, giving the familiar

$$U = (1/2)m(d\mathbf{r}/dt)^2 \tag{11}$$

of nonrelativistic physics.

B. Lorentz Invariance

On the other hand, according to Lorentz invariance, the conservation law (4) for the center of energy of this one-particle system becomes

$$\left(\mathbf{F}{\cdot}\frac{d\mathbf{r}}{dt}\right)\mathbf{r} = \frac{d}{dt}\left(U\mathbf{r}\right) - c^2\mathbf{p} \tag{12}$$

Expanding the time derivative $d(U\mathbf{r})/dt = (dU/dt)\mathbf{r} + U(d\mathbf{r}/dt)$ in Eq (12), noting the power relation $\mathbf{F} \cdot (d\mathbf{r}/dt) = dU/dt$ in Eq. (8), and cancelling two terms, we have

$$0 = U\frac{d\mathbf{r}}{dt} - c^2\mathbf{p} \tag{13}$$

However, we can then eliminate $d\mathbf{r}/dt$ between $dU/dt = (d\mathbf{p}/dt) \cdot (d\mathbf{r}/dt)$ from Eqs. (7) and (8) and $U(d\mathbf{r}/dt) = c^2\mathbf{p}$ in Eq. (13) to obtain

$$\frac{dU}{dt} = \frac{d\mathbf{p}}{dt} \cdot \frac{c^2\mathbf{p}}{U} \tag{14}$$

which has the solution $U^2 = c^2p^2 + const$. If we choose the constant as m^2c^4, then we have the familiar relativistic expression for particle mechanical energy

$$U = (c^2p^2 + m^2c^4)^{1/2} \tag{15}$$

where m is the particle restmass. Also, we can use Eq. (15) to eliminate the energy U in Eq. (13) and solve for $\mathbf{p}$ to obtain the familiar relativistic expression for the linear momentum

$$\mathbf{p} = m\gamma(d\mathbf{r}/dt) \tag{16}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ with $\mathbf{v} = d\mathbf{r}/dt$ and $v = |\mathbf{v}|$.

IV.   RELATIVISTIC MECHANICS OF PARTICLE COLLISIONS

A.   Familiar Particle Collisions at Points

Particle collisions at points are familiar textbook subjects in both nonrelativistic and relativistic mechanics. In these cases, there are no external forces and the particle momenta and energies correspond to the mechanical energy and momentum given by Eqs. (10) and (11) or by Eqs. (15) and (16) for the nonrelativistic and relativistic cases respectively. The collisions are assumed to conserve total energy and total momentum. Since the particle collisions occur at a single point $\mathbf{r}$, the conservation laws of energy and momentum are sufficient to guarantee the additional laws of angular momentum conservation, and constant motion of the center of mass in the Galilean case or constant motion of the center of (mechanical) energy in the Lorentz case. In these cases, the conservation laws can be applied in any inertial frame and will be found to hold in any other inertial frame.

1.   Relativistic Collision

Here we introduce an illustration of elastic collisions at a point; later we will reinvestigate the collision when interaction potentials are introduced. For simplicity, we consider two particles m_1 and m_2 of equal mass $m = m_1 = m_2$, the first of which is approaching with speed v along the negative x-axis, $x_1 = vt$ for $t < 0$ and the second of which is initially at rest at the coordinate origin, $x_2 = 0$ for $t < 0$. Thus before the collision, the system energy times the center of energy is given by $(m\gamma c^2 + mc^2)\mathcal{X}_{energy} = (m\gamma c^2)vt + mc^2 0$ so that the center of energy moves with constant velocity

$$\mathcal{X}_{energy} = \frac{(m\gamma c^2)vt}{(m\gamma c^2 + mc^2)} \tag{17}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. We assume that the two particles have no associated potential energy between them so that the particles will not interact until the point collision at the origin at time $t = 0$. At this collision, the particles exchange energy and momentum. After the collision, the first particle comes to rest at the origin while the second particle carries the energy $m\gamma c^2$ along the positive x-axis. The center of energy is given by $(mc^2 + m\gamma c^2)\mathcal{X}_{energy} = mc^2 0 + (m\gamma c^2)vt$, so that the motion is again given by Eq. (17). In this collision at a single

point, the system center of energy moves with constant velocity, and the fourth relativistic conservation law (4) is indeed satisfied.

2. Nonrelativistic Collision

We could also treat this collision problem in nonrelativistic physics. In this case the fourth conservation law involves the center of restmass following from $(m + m)\mathcal{X}_{mass} = mvt + m0$,

$$\mathcal{X}_{mass} = \frac{mvt}{2m} \tag{18}$$

After the collision at the origin, the center or restmass is given by $(m+m)\mathcal{X}_{mass} = m0+mvt$, which again leads to Eq. (18). Thus the center of restmass moves with constant velocity when nonrelativistic expressions are used for the mechanical energy and momentum.

3. Mixed Relativistic-Nonrelativistic Collision

It is also possible to consider the elastic collisions of point particles even when one colliding particle is described by nonrelativistic energy and momentum and the other by relativistic energy and momentum. Within a single inertial frame, the momentum, energy, and angular momentum conservation laws can all be handled satisfactorily. Thus, for our simple example, we can describe the energy and momentum of the first (incoming) particle by relativistic expressions $U_1 = m\gamma c^2$ and $p_1 = m\gamma v$ while using nonrelativistic expressions for the second particle. In this case, the first particle would not be brought to rest on collision with the particle of equal mass which had been sitting at the origin. Rather, we would solve for the final velocities of both particles using the momentum and energy conservation laws

$$m\gamma v + 0 = m\gamma_1 v_1 + mv_2 \tag{19}$$

and

$$m\gamma c^2 + 0 = m\gamma_1 c^2 + \frac{1}{2}mv_2^2 \tag{20}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\gamma_1 = (1 - v_1^2/c^2)^{-1/2}$. We have two equations in the two unknown final velocities v_1 and v_2, and so can solve for these quantities in terms of the initial incoming particle velocity v.[6] Our example will satisfy neither constant motion of the center of energy nor constant motion of the center of restmass. Such a situation of

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mixed relativistic and nonrelativistic expressions involves neither Galilean invariance nor Lorentz invariance, and the collision outcome will depend upon the specific inertial frame in which the conservation laws for the collision are applied. Since most physicists are fully aware of only the first three conservation laws, the failure of the fourth conservation law on mixing Galilean- and Lorentz-invariant systems (usually nonrelativistic mechanics combined with electrodynamics) is rarely noted.[7]

B. Particle Collisions Involving Potentials

Although point collisions within nonrelativistic or relativistic mechanics are quite satisfactory, the introduction of potentials between relativistic particles is quite another matter. Indeed, the no-interaction theorem of Currie, Jordan, and Sudarshan[2] states that within relativistic mechanics the use of such potentials is completely forbidden by our usual ideas of mechanics. The Lorentz-invariant center-of-energy requirement of Eq. (4) is so restrictive that it does not allow mechanical interactions through potentials. In order to escape the no-interaction theorem, one must turn to a field theory.

1. Collision Through a Rigid Pole

Here we illustrate what is involved in the no-interaction theorem by reconsidering the collision of our two particles. Let us now consider the same collision of two relativistic particles but this time the first particle (at $x_1 = vt$ for early times) carries a massless rigid pole of length R which precedes the particle. This rigid pole produces a collision with the second particle when there is still a separation R between them. Now the collision occurs at time $t = -R/v$. The first particle stops at $x_1 = -R$ while its energy and momentum are transferred instantaneously to the second particle which moves as $x_2 = v(t + R/v)$. In this case, the system center of energy does not move with constant velocity, but rather makes a discontinuous jump as the energy is transferred instantaneously along the pole to the other particle. Before the collision, the center of energy motion is given by Eq. (17), but after the collision, the center of energy motion follows from $(mc^2 + m\gamma c^2)\mathcal{X}_{energy} = mc^2(-R) + m\gamma c^2 v(t + R/v)$ giving

$$\mathcal{X}_{energy} = \frac{(m\gamma c^2)vt + mc^2(\gamma - 1)R}{(mc^2 + m\gamma c^2)} \tag{21}$$

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which does not agree with Eq. (17).

Since such rigid-pole collisions do not satisfy the conservation laws associated with Lorentz-invariant behavior, they are not allowed in relativistic theory. Indeed in kinematic discussions of special relativity, students are regularly warned against the possibility of rigid poles. However, we should emphasize that such rigid poles do allow the much-less-stringent condition (6) of Galilean invariance which corresponds to the continuous flow of rest mass. We saw above that if we treat the two-particle collision using nonrelativistic physics, then the center of restmass $\mathcal{X}_{mass}$ moves with the constant velocity of Eq. (18) before the collision. However, the center of restmass also moves with constant velocity even after the two nonrelativistic particles collide through a rigid pole. Thus after the collision, the first particle comes to rest at $-R$ while the second particle moves off from the origin with a position given by $x_2 = v(t + R/v)$ since the collision occurred at time $t = -R/v$. Thus the center of restmass is calculated as $(m+m)\mathcal{X}_{mass} = m(-R) + m[v(t+R/v)]$ which gives exactly the same result as in Eq. (18) which held before the collision. The rigid pole transfers energy, not restmass, and so gives a discontinuous jump in the center of energy but not in the center of restmass.

2. *Collision Through a General Potential $V(|x_2 - x_1|)$*

The use of a rigid pole in the collision of our example is equivalent to considering a hard-sphere potential between the particles. Our example illustrates that a hard-sphere potential leads to an instantaneous transfer of energy and hence to a failure of the Lorentz-invariant conservation law requiring a constant velocity for the center of energy in an isolated system. The use of a hard-sphere potential is an extreme situation for a potential. One might hope that by use of a softer interaction (perhaps through massless springs) one might be able to accommodate potentials into relativistic mechanics. However, this can not be done in general. The same basic flaw keeps reappearing; there is a transfer of energy between spatial points without a continuous energy flow through the intervening space.

We can illustrate this difficulty by considering the collision along the x-axis of two particles m_1 and m_2 of relativistic energy and momentum interacting through a general potential $V(|x_2 - x_1|)$ dependent upon the separation between the particles where we assign the center of energy associated with the potential energy to a location half-way between the

particles. In this case the energy times the center of energy takes the form

$$U\mathcal{X}_{energy} = m_1\gamma_1c^2x_1 + m_2\gamma_2c^2x_2 + V(|x_2 - x_1|)\frac{(x_1 + x_2)}{2} \tag{22}$$

We require that the particles experience forces associated with Newton's second law arising from the potential $V(|x_2 - x_1|)$ and for simplicity assume $x_1 < x_2$ so that $|x_2 - x_1| = x_2 - x_1$. Then the expressions (7) and (8) for Newton's second law and particle mechanical energy change take the form

$$\frac{dp_1}{dt} = V'(|x_2 - x_1|) \quad \text{and} \quad \frac{dp_2}{dt} = -V'(|x_2 - x_1|) \tag{23}$$

and

$$\frac{d}{dt}(m_1\gamma_1c^2) = \frac{dp_1}{dt}v_1 \quad \text{and} \quad \frac{d}{dt}(m_2\gamma_2c^2) = \frac{dp_2}{dt}v_2 \tag{24}$$

where $V'(|x_2 - x_1|)$ refers to the derivative of the potential with respect to its argument. Then the total energy $U = m_1\gamma_1c^2 + m_2\gamma_2c^2 + V(|x_2 - x_1|)$ and momentum $P = p_1 + p_2$ are both constant in time and

$$\begin{aligned}
\frac{d}{dt}(U\mathcal{X}_{energy}) = U\frac{d\mathcal{X}_{energy}}{dt} &= \frac{d}{dt}\left(m_1\gamma_1c^2x_1 + m_2\gamma_2c^2x_2 + V(|x_2 - x_1|)\frac{(x_1 + x_2)}{2}\right) \\
&= \frac{d}{dt}(m_1\gamma_1c^2)x_1 + \frac{d}{dt}(m_2\gamma_2c^2)x_2 + V'(|x_2 - x_1|)(v_2 - v_1)\frac{(x_1 + x_2)}{2} \\
&+ m_1\gamma_1c^2v_1 + m_2\gamma_2c^2v_2 + V(|x_2 - x_1|)\frac{(v_1 + v_2)}{2} \\
&= \left(\frac{dp_1}{dt} - V'(|x_2 - x_1|)\right)v_1x_1 + \left(\frac{dp_2}{dt} - V'(|x_2 - x_1|)\right)v_2x_2 + c^2(p_1 + p_2) \\
&+ V(|x_2 - x_1|)\frac{(v_1 + v_2)}{2} + \frac{1}{2}V'(|x_2 - x_1|)(v_1 + v_2)(x_1 - x_2) \\
&= c^2P + [V(|x_2 - x_1|) - V'(|x_2 - x_1|)(x_2 - x_1)]\frac{(v_1 + v_2)}{2} \tag{25}
\end{aligned}$$

where we have used the equations of motion to eliminate two terms in round brackets. Since x_1, x_2, v_1, and v_2 can be chosen arbitrarily, the only way to satisfy the fourth Lorentz-invariant conservation law is to have the square parenthesis in the last line of Eq. (25) vanish. This requires that

$$V(|x_2 - x_1|) = k|x_2 - x_1| \tag{26}$$

where k is a constant, and corresponds to a constant force between the particles. In this case, there is a smooth rather than a sudden transfer of energy between the particles. However, this unique potential giving Lorentz-invariant behavior for the center of energy seems

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surprising as an interparticle potential since it increases in magnitude with distance rather than decreases. There is no asymptotic region where the particles can be regarded as unaffected by the other particle. Actually, this unique potential is quite familiar, not as the potential between two particles but rather as the electrostatic potential energy for two uniformly charged parallel plates of large area $A = L \times L$ and small separation $|x_2 - x_1| << L$. In this case, the uniform electrostatic field between the plates indeed contributes an energy whose natural center of energy is halfway between the plates. Here the transfer of energy between the field and the plate is carried out locally at each plate, with the field energy either appearing or disappearing as the plate mechanical energy decreases or increases. This parallel-plate example has been discusses in more detail in two other analyses.[8] The *interparticle* potential which leads to the potential (24) between the (*multiparticle*) parallel plates is the Coulomb potential between charged particles, which is part of classical electrodynamic field theory and is not part of a Lorentz-invariant mechanical system of particles with potentials. The no-interaction theorem of Currie, Jordan, and Sudarshan[2] applies to Lorentz-invariant mechanical systems and does not apply to field theories.

V. NONRELATIVISTIC LAGRANGIANS FOR PARTICLES AND LORENTZ-INVARIANT EXTENSION TO ORDER v^2/c^2

Although the no-interaction theorem of relativistic mechanics indicates that it is hopeless to create a conventional fully relativistic mechanical theory with an arbitrary potential, one might try to extend nonrelativistic mechanics toward a theory which is approximately relativistic to higher orders in v/c, where v is the typical particle velocity in some inertial frame. This would give us some indication of just when we need to turn to field theory in order to obtain a fully relativistic system.

The nonrelativistic system of two particles interacting through a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ can be described by a Lagrangian

$$L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{27}$$

The invariance of this Lagrangian under spacetime translations and spatial rotations leads to the conservation laws for linear momentum, energy, and angular momentum. In order to extend this system toward an approximately Lorentz-invariant system, we must preserve

the invariance of the Lagrangian under spacetime translations and spatial rotations while adding (small) additional space-dependent and velocity-dependent functions of order v^2/c^2.

A. Lagrangian Lorentz-Invariant Through Order v^2/c^2

Since the energy and momentum of an isolated system should form a Lorentz four-vector, we expect the potential energy $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ to be related to momentum in a different inertial frame. Thus in a moving frame, we expect to find velocity-dependent forces between the particles in addition to the position-dependent forces found in the original frame. If we require Lorentz invariance through order v^2/c^2, then the velocity-dependent terms must appear in the Lagrangian in any inertial frame. By working backwards from the requirement of Lorentz invariance given in Eq. (4) and requiring that the condition hold through order v^2/c^2, we find that the Lagrangian extended from the nonrelativistic expression can be written as[9]

$$
\begin{aligned}
L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = {} & -m_1 c^2 (1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2} - m_2 c^2 (1 - \dot{\mathbf{r}}_2^2/c^2)^{1/2} - V(|\mathbf{r}_1 - \mathbf{r}_2|) \\
& + \frac{1}{2} V(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2}{c^2} - \frac{1}{2} V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \dot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{c^2 |\mathbf{r}_1 - \mathbf{r}_2|}
\end{aligned}
\tag{28}
$$

where we have introduced the Lagrangian terms leading to relativistic mechanical momentum and energy for the particles, and where $V'(|\mathbf{r}_1 - \mathbf{r}_2|)$ refers to the derivative of the potential function with respect to its argument.

We can check the Lorentz invariance of this Lagrangian through order v^2/c^2 by showing that Eq. (4) (with $\mathbf{F}_{ext\,i} = 0$) holds through this order; in other words, the system center of energy moves with constant velocity through order v^2/c^2. The system energy U times the center of energy of the system $\overrightarrow{X}$ through zero-order in v/c is given by

$$
U\overrightarrow{X} = m_1 (c^2 + \frac{1}{2}\dot{\mathbf{r}}_1^2)\mathbf{r}_1 + m_2(c^2 + \frac{1}{2}\dot{\mathbf{r}}_2^2)\mathbf{r}_2 + V(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 + \mathbf{r}_2)}{2}
\tag{29}
$$

corresponding to the restmass energy and kinetic energy of the two particles located at their respective positions $\mathbf{r}_1$ and $\mathbf{r}_2$ plus the interaction potential energy located half way between the positions of the two particles. Since the Lagrangian in Eq. (28) has no explicit time dependence, the system energy U is constant in time. Taking the time derivative of Eq.

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(29), we find

$$\frac{d}{dt}(U\overrightarrow{X}) = U\frac{d\overrightarrow{X}}{dt} = m_1(c^2 + \frac{1}{2}\dot{\mathbf{r}}_1^2)\dot{\mathbf{r}}_1 + m_2(c^2 + \frac{1}{2}\dot{\mathbf{r}}_2^2)\dot{\mathbf{r}}_2 + (m_1\ddot{\mathbf{r}}_1\cdot\mathbf{r}_1)\mathbf{r}_1 + (m_2\ddot{\mathbf{r}}_2\cdot\mathbf{r}_2)\mathbf{r}_2$$
$$+ \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2) + \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\cdot(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 + \mathbf{r}_2) \qquad (30)$$

It is sufficient to use the nonrelativistic equations of motion,

$$m_1\ddot{\mathbf{r}}_1 = -V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad (31)$$

$$m_2\ddot{\mathbf{r}}_2 = V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad (32)$$

to transform Eq. (30) into the form

$$\frac{d}{dt}(U\overrightarrow{X}) = U\frac{d\overrightarrow{X}}{dt} = m_1(c^2 + \frac{1}{2}\dot{\mathbf{r}}_1^2)\dot{\mathbf{r}}_1 + m_2(c^2 + \frac{1}{2}\dot{\mathbf{r}}_2^2)\dot{\mathbf{r}}_2$$
$$- \left(V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\cdot\dot{\mathbf{r}}_1\right)\mathbf{r}_1 + \left(V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\cdot\dot{\mathbf{r}}_2\right)\mathbf{r}_2$$
$$+ \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2) + \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\cdot(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 + \mathbf{r}_2) \qquad (33)$$

The momenta can be obtained from the Lagrangian in Eq. (28) as

$$\mathbf{p}_1 = \frac{\partial L}{\partial\dot{\mathbf{r}}_1} = m_1\dot{\mathbf{r}}_1\left(1 - \frac{\dot{\mathbf{r}}_1^2}{c^2}\right)^{-1/2} + \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\dot{\mathbf{r}}_2}{c^2}$$
$$- \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\dot{\mathbf{r}}_2}{c^2}\cdot(\mathbf{r}_1 - \mathbf{r}_2) \qquad (34)$$

$$\mathbf{p}_2 = \frac{\partial L}{\partial\dot{\mathbf{r}}_2} = m_2\dot{\mathbf{r}}_2\left(1 - \frac{\dot{\mathbf{r}}_2^2}{c^2}\right)^{-1/2} + \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\dot{\mathbf{r}}_1}{c^2}$$
$$- \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\dot{\mathbf{r}}_1}{c^2}\cdot(\mathbf{r}_1 - \mathbf{r}_2) \qquad (35)$$

giving total linear momentum

$$\mathbf{P} = m_1\dot{\mathbf{r}}_1\left(1 - \frac{\dot{\mathbf{r}}_1^2}{c^2}\right)^{-1/2} + m_2\dot{\mathbf{r}}_2\left(1 - \frac{\dot{\mathbf{r}}_2^2}{c^2}\right)^{-1/2}$$
$$+ \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\dot{\mathbf{r}}_1}{c^2} + \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\dot{\mathbf{r}}_2}{c^2}$$
$$- \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\dot{\mathbf{r}}_1}{c^2}\cdot(\mathbf{r}_1 - \mathbf{r}_2) - \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\dot{\mathbf{r}}_2}{c^2}\cdot(\mathbf{r}_1 - \mathbf{r}_2) \qquad (36)$$

Comparing Eqs. (30) and (36) after reorganizing a few terms and expanding any factors of $\gamma = (1 - v^2/c^2)^{-1/2}$, we find that indeed Eq. (4) holds. The system of Eq. (28) is indeed Lorentz invariant through order v^2/c^2.

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We notice from Eqs. (34)-(36) that momentum is now no longer connected exclusively with restmass times velocity, as is required by the fourth conservation law for Galilean invariance. The center of restmass for the system of Eq. (28) no longer moves with constant velocity. Now linear momentum is associated with the parameters of the potential energy, and the system is being transformed toward a field-theory point of view.

B. Velocity-Dependent and Acceleration-Dependent Forces in Lorentz-Invariant Systems Through Order v^2/c^2

The Lagrange equations of motion follow from the Lagrangian in Eq. (28); for the particle at $\mathbf{r}_1$, the equation takes the form

$$
\begin{aligned}
0 = {} & \frac{d}{dt}\left(\frac{m_1\dot{\mathbf{r}}_1}{(1-\dot{\mathbf{r}}_1^2/c^2)^{1/2}} + \frac{1}{2}V(|\mathbf{r}_1-\mathbf{r}_2|)\frac{\dot{\mathbf{r}}_2}{c^2} - \frac{1}{2}V'(|\mathbf{r}_1-\mathbf{r}_2|)\frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}\frac{\dot{\mathbf{r}}_2}{c^2}\cdot(\mathbf{r}_1-\mathbf{r}_2)\right) \\
& - \frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}V'(|\mathbf{r}_1-\mathbf{r}_2|)\left(-1+\frac{\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_2}{2c^2}\right) + \frac{\dot{\mathbf{r}}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\dot{\mathbf{r}}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)}{2c^2\,|\mathbf{r}_1-\mathbf{r}_2|^2} \\
& + \frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}V''(|\mathbf{r}_1-\mathbf{r}_2|)\frac{\dot{\mathbf{r}}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\dot{\mathbf{r}}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)}{2c^2\,|\mathbf{r}_1-\mathbf{r}_2|^2} \\
& + V'(|\mathbf{r}_1-\mathbf{r}_2|)\left(\frac{\dot{\mathbf{r}}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\dot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)\dot{\mathbf{r}}_1}{2c^2\,|\mathbf{r}_1-\mathbf{r}_2|}\right)
\end{aligned}
\tag{37}
$$

The equations of motion can be rewritten as forces acting on the particles to change the mechanical momentum. For the particle at $\mathbf{r}_1$, this becomes

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{m\dot{\mathbf{r}}_1}{(1-\dot{\mathbf{r}}_1^2/c^2)^{1/2}}\right) = {} & -V'(|\mathbf{r}_1-\mathbf{r}_2|)\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}\left[1 + \frac{1}{2}\left(\frac{\dot{\mathbf{r}}_2}{c}\right)^2\right] \\
& - \frac{\mathbf{r}_1-\mathbf{r}_2}{2c^2\,|\mathbf{r}_1-\mathbf{r}_2|}\left(\frac{V''(|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|} - \frac{V'(|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|^2}\right)[\dot{\mathbf{r}}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)]^2 \\
& - \frac{1}{2c^2}\left(V(|\mathbf{r}_1-\mathbf{r}_2|)\ddot{\mathbf{r}}_2 - V'(|\mathbf{r}_1-\mathbf{r}_2|)\frac{[\ddot{\mathbf{r}}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)](\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^2}\right) \\
& - \frac{\dot{\mathbf{r}}_1}{c}\times\left(\frac{\dot{\mathbf{r}}_2}{c}\times\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}V'(|\mathbf{r}_1-\mathbf{r}_2|)\right)
\end{aligned}
\tag{38}
$$

We notice that the force on the first particle involves not only the force arising from the original nonrelativistic potential function, but also forces depending upon the velocities of both particles and upon the acceleration of the other particle. These forces were not part of the original nonrelativistic theory. Such forces are absent from the accounts in the mechanics textbooks[10] and from the articles which treat "relativistic" motion for a single

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particle. The single particle appearing in the Lagrangian of these treatments actually produces velocity-dependent and acceleration-dependent forces back on the prescribed sources whose momentum and energy are never discussed.

The most famous Lagrangian which is Lorentz invariant through v^2/c^2 is that obtained from the Coulomb potential $V(|\mathbf{r}_1 - \mathbf{r}_2|) = q_1 q_2/|\mathbf{r}_1 - \mathbf{r}_2|$. In this case the Lagrangian of Eq. (28) becomes

$$L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = -m_1 c^2 (1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2} - m_2 c^2 (1 - \dot{\mathbf{r}}_2^2/c^2)^{1/2} - \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$
$$+ \frac{1}{2} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2}{c^2} + \frac{1}{2} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \dot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{c^2 |\mathbf{r}_1 - \mathbf{r}_2|^2} \tag{39}$$

If in Eq. (39) we expand the free-particle expressions $-mc^2(1 - \dot{\mathbf{r}}^2/c^2)^{1/2}$ through second order in v/c, then this becomes the Darwin Lagrangian which sometime appears in electromagnetism textbooks as an approximation to the interaction of charged particles.[11] The approximation is an accurate description of the classical electromagnetic interaction between charged particles through second order in v/c for small separations between the particles. The Lagrangian equation of motion following from Eq.(39) becomes (for the particle at position $\mathbf{r}_1$)

$$\frac{d}{dt} \left(\frac{m_1 \dot{\mathbf{r}}_1}{(1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2}} \right) = q_1 [q_2 \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \left\{ 1 + \frac{1}{2} \left(\frac{\dot{\mathbf{r}}_2}{c} \right)^2 - \frac{3}{2} \left(\frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot \dot{\mathbf{r}}_2}{c|\mathbf{r}_1 - \mathbf{r}_2|} \right)^2 \right\}$$
$$- \frac{q_2}{2c^2} \left(\ddot{\mathbf{r}}_2 + \frac{[\ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)](\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \right)]$$
$$+ q_1 \frac{\dot{\mathbf{r}}_1}{c} \times \left[q_2 \frac{\dot{\mathbf{r}}_2}{c} \times \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right] \tag{40}$$

where we have rewritten the Lagrangian equation in the form $d\mathbf{p}_1/dt = q_1 \mathbf{E} + q_1(\dot{\mathbf{r}}_1/c) \times \mathbf{B}$ with $\mathbf{p}_1$ the mechanical particle momentum.[12] The velocity- and acceleration-dependent forces in Eq.(40) correspond to fields arising from electromagnetic induction. In the textbooks, electromagnetic induction is always treated without reference to any charged particles which may be producing the induction fields, a very different point of view from that which follows from the Darwin Lagrangian.

Contemporary physics regards special relativity as a metatheory to which (locally) all theories describing nature should conform. Thus in nonrelativistic classical mechanics, there is the unspoken implication that the nonrelativistic interaction between point particles at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ under a general potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ is the small-velocity limit of some

fully relativistic theory of interacting point particles which might occur in nature. However, the use of a general potential can be misleading for both students and researchers. Here we demonstrate that an arbitrary nonrelativistic potential function can indeed be extended to a Lagrangian which is Lorentz-invariant through order v^2/c^2; however, the extension requires the introduction of velocity-dependent and acceleration-dependent forces which go unmentioned in the mechanics textbooks.

VI. DISCUSSION

In this paper, we have tried to give simple examples which illustrate the conservation laws of Lorentz-invariant systems and which suggest the reason for the "no-interaction theorem" of relativistic mechanics beyond point binary collisions. The introduction of position-dependent potential energy places a burden on the fourth relativistic conservation law requiring the constant velocity of the center of energy. In general this burden is so restrictive that there can be no interaction through a potential without an accompanying field which carries both energy and linear momentum.

The treatments of relativistic mechanics in textbooks can be misleading. In mechanics text books, we often find discussions of single-particle motion in potentials.[10] The most famous (and appropriate example) is the motion of an particle in a Coulomb potential. This can be regarded as an approximation to the fully relativistic two-charged-particle electromagnetic interaction where the nucleus is much more massive than the orbiting electron, and so the classical electrodynamic analysis agrees with the one-particle mechanical analysis in the approximation which ignores emission and/or absorption by the electromagnetic radiation field. However, there is also a mechanics text book which discusses a particle with relativistic momentum in a harmonic oscillator potential, with no mention of field theory.[13] An uninformed student might easily assume that this is an example of a relativistic interaction between two particles, one of which is much more massive than the other, which are interacting through the harmonic oscillator potential. We have emphasized that the no-interaction theorem indicates that such a purely mechanical relativistic interaction is not possible.

Classical electromagnetism is a relativistic theory which was developed during the nineteenth century before the ideas of special relativity. Indeed, special relativity arose at

the beginning of the twentieth century as a response to the conflict of electromagnetism with nonrelativistic mechanics. Around the same time, quantum mechanics was introduced in response to the mismatch between electromagnetic radiation equilibrium (blackbody radiation) and classical statistical mechanics (which is based on nonrelativistic mechanics). Although quantum theory and special relativity have gone on to enormous successes, they have left behind a number of unresolved questions within classical physics. For example, the blackbody radiation problem has never been solved within relativistic classical physics.[14] There have been discussions of classical radiation equilibrium using nonrelativistic mechanical scatterers and even one calculation of a scattering particle using relativistic mechanical momentum in a general class of non-Coulomb potentials.[15] However, there has never been a treatment of scattering by a relativistic particle in the Coulomb potential of classical electrodynamics, despite the fact that the Coulomb potential has all the qualitative aspects which might allow classical radiation equilibrium at a spectrum with finite thermal energy.

We conclude that misconceptions regarding potentials which can be regarded as approximations to relativistic systems are relevant for treatments in mechanics textbooks and perhaps also for the description of nature within classical theory.

Acknowledgement

[1] This belief is encouraged by mechanics texts which write one-particle "relativistic Lagrangians" involving arbitrary potential functions $V(\mathbf{r})$ rather than simply applying external forces to a relativistic particle. See, for example, H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics 3rd ed.* (Addison-Wesley, New York 2002), p. 313, or J. V. Jose and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press 1998), p. 210.

[2] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, "Relativistic Invariance and Hamiltonian Theories of Interacting Particles," Rev. Mod. Phys. **34**, 350-375 (1963).

[3] See pages 324 and 353 of the text by Goldstein, Poole, and Safko in ref. 1.

[4] S. Coleman and J.H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys.

Rev. **171**, 1370-1375 (1968).

[5] T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy," Am. J. Phys. **73**, 953-961 (2005).

[6] Indeed there have been numerical calculations of the energy distributions for colliding particles in a reflecting-walled box; it turns out that the nonrelativistic particle takes on the Maxwell velocity distribution and the relativistic particle takes on the Jüttner distribution in the rest frame of the box.[T.H. Boyer unpublished]

[7] However, this is pointed out emphatically by F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA 1965), p. 210.

[8] See ref. 5 and also T. H. Boyer, "Relativistic Mechanics and a Special Role for the Coulomb Potential," arXiv.org (physics/0810.0434).

[9] Approximately Lorent-invariant Lagrangians of this form appear in the work of P. Havas and J. Stachel, "Invariances of Approximately Relativistic Lagrangians and the Center-of-Mass Theorem. I," Phys. Rev. **185**, 1636-1647 (1969), Eq. (46) and F. J. Kennedy, "Approximately Relativistic Interactions," Am. J. Phys. **40**, 63-74 (1972), Eq. (27). See also H. W. Woodcock and P. Havas, "Approximately Relativistic Lagrangians for Classical Interacting Point Particles," Phys. Rev. D **6**, 3422-3444 (1972).

[10] See Sections 7.9, 7.10 of the text by Goldstein, Poole, and Safko in ref. 1, and pages 209-212 of the text by Jose and Slaetan in ref. 1..

[11] J. D. Jackson, *Classical Electrodynamics 3rd ed* (Wiley, New York 1999), p. 596-598; the Darwin Lagrangian is given in Eq. (12.82).

[12] The expressions for the electric and magnetic fields agree with those given by L. Page and N.I. Adams, "Action and Reaction Between Moving Charges," Am. J. Phys. **13**, 141-147 (1945).

[13] See Section 7.9, p. 316, of the text by Goldstein, Poole, and Safko in ref. 1. Also see Section 6-14, pp. 179-181 of Rohrich's text in ref. 7 where the harmonic oscillator potential is used.

[14] See, for example, the discussions by T. H. Boyer, "Blackbody radiation, conformal symmetry, and the mismatch between classical mechanics and electromagnetism," J. Phys. A: Math. Gen. **38**, 1807-1821 (2005); "Connecting blackbody radiation, relativity, and discrete charge in classical electrodynamics," Found. Phys. **37**, 999-1026 (2007).

[15] The mistaken idea of Lorentz-invariant behavior as a description of nature involving only the use of relativistic mechanical momentum appears in the theoretical analysis of black-

body radiation by R. Blanco, L. Pesquera, and E. Santos, "Equilibrium between radiation and matter for classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution from Rayleigh-Jeans spectrum," Phys. Rev. D **27**, 1254-1287 (1983), and "Equilibrium between radiation and matter for classical relativistic multiperiodic systems. II. Study of radiative equilibrium with Rayleigh-Jeans radiation," *ibid.* **29**, 2240-2254 (1984).